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The Discrete Series of $Sp(n, \mathbb{R})$

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Abstract

For each irreducible representation $[\lambda]$ of $U(n)$, a discrete irreducible unitary representation of $Sp(n)$ is constructed for which the oscillator Hamiltonian is bounded from below with its ground-state eigenspace transforming according to $[\lambda]$ under $U(n)$. A basis of eigenstates for the harmonic oscillator is determined and the action of the Lie algebra $sp(n)$ on that basis explicitly given. Connections with the Bohr collective vibrational model are established.

1. *Introduction*

The real symplectic group $Sp(n) = Sp(n, \mathbb{R})$ is the noncompact simple Lie group of dimension $n(2n + 1)$ given by the linear transformations preserving a skew-symmetric bilinear form on a $2n$-dimensional real vector space:

$$Sp(n) = \{g \in M_{2n}(\mathbb{R}) \mid g^t J g = J\} \tag{1.1}$$

where $M_{2n}(\mathbb{R})$ denotes the set of $2n \times 2n$ real matrices, elements of which are typically written

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, \qquad g_i \in M_n(\mathbb{R})$$

and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$$

It plays a basic role in the qualitative theory of Hamiltonian systems (Abraham, 1967) as the group of linear canonical transformations on $2n$-dimensional phase space (Moshinsky and Quesne, 1971). Moreover, $Sp(n)$ is the unique dynamical group containing $SU(n)$ among the classical groups (Hwa and Nuyts, 1966; Mukunda et al., 1965; Moshinsky and Quesne, 1970).

In this article a discrete series of unitary representations of $Sp(n)$ is constructed. Previously we have given the principal series for $Sp(n)$ (Rosensteel and Rowe, 1975a).

Each of the unitary representations of $Sp(n)$ defines a skew-adjoint representation of the real Lie algebra $sp(n)$ of the group $Sp(n)$:

$$sp(n) = \left\{ x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1{}^t \end{pmatrix} \in M_{2n}(\mathbb{R}) \,\middle|\, x_2{}^t = x_2, x_3{}^t = x_3 \right\} \tag{1.2}$$

The complexification of $sp(n)$ is denoted by C_n in the Cartan classification of complex simple Lie algebras (Jacobson, 1962).

For applications to many-body physics, the interest in the Lie algebra $sp(n)$ is through its isomorphism to $bl(n)$, the real Lie algebra of operators spanned by the skew-adjoint one-body bilinear products in the position $X_{j\alpha}$ and momentum $P_{j\alpha}$ observables for an N-particle system in an n-dimensional Euclidean space, $j = 1, 2, \ldots, N$, $\alpha = 1, 2, \ldots, n$. A basis for $bl(n)$ is given by

$$\begin{aligned} iL_{\alpha\beta} &= i \sum_{j=1}^{N} (X_{j\alpha}P_{j\beta} - X_{j\beta}P_{j\alpha}) \\ iS_{\alpha\beta} &= i \sum_{j=1}^{N} (X_{j\alpha}P_{j\beta} + X_{j\beta}P_{j\alpha} - i\delta_{\alpha\beta}I) \\ iQ_{\alpha\beta} &= i \sum_{j=1}^{N} X_{j\alpha}X_{j\beta} \\ iK_{\alpha\beta} &= i \sum_{j=1}^{N} P_{j\alpha}P_{j\beta} \end{aligned} \tag{1.3}$$

To establish the isomorphism of $sp(n)$ with $bl(n)$, define the real vector space V_n of self-adjoint operators spanned by

$$X_\alpha = \sum_{j=1}^{N} X_{j\alpha}$$

and

$$P_\alpha = \sum_{j=1}^{N} P_{j\alpha}$$

$$V_n = \left\{ \zeta = \sum_{\alpha=1}^{n} (\zeta_\alpha X_\alpha + \zeta_{\alpha+n} P_\alpha),\ \zeta_\alpha \in \mathbb{R} \right\} \tag{1.4}$$

Then the endomorphism of V_n given by

$$\zeta \to e^s \zeta e^{-s} \qquad \text{for } s \in bl(n), \zeta \in V_n \tag{1.5}$$

is a symplectic transformation with respect to the skew-symmetric form on V_n:

$$(\zeta, \zeta') = -\left(\frac{i}{N}\right) \frac{\langle \phi | [\zeta, \zeta'] \phi \rangle}{\langle \phi | \phi \rangle} \qquad \text{for } \zeta, \zeta' \in V_n \tag{1.6}$$

for any wave function $\phi \neq 0$. Since $[\zeta, \zeta']$ is proportional to the identity operator for $\zeta, \zeta' \in V_n$, this form is independent of ϕ.

The symplectic transformation of equation (1.5) defines a representation of the Lie algebra $bl(n)$ on V_n by

$$\zeta \to [s, \zeta] \qquad \text{for } s \in bl(n), \zeta \in V_n \tag{1.7}$$

Then the isomorphism of $bl(n)$ with $sp(n)$ is given by the above action with the identification of V_n with $\mathbb{R}^{2n}$, the space on which $sp(n)$ acts, given by

$$\begin{aligned} X_\alpha &\to e_\alpha \\ & \qquad\qquad \alpha = 1, 2, \ldots, n \\ P_\alpha &\to e_{\alpha+n} \end{aligned} \tag{1.8}$$

where $e_\mu \in \mathbb{R}^{2n}$ denotes the column vector whose only nonzero entry is one in the μ row. The explicit isomorphism is

$$\begin{aligned} iL_{\alpha\beta} &\to (E_{\alpha\beta} - E_{\beta\alpha}) + (E_{\alpha+n,\beta+n} - E_{\beta+n,\alpha+n}) \\ iS_{\alpha\beta} &\to (E_{\alpha\beta} + E_{\beta\alpha}) - (E_{\alpha+n,\beta+n} + E_{\beta+n,\alpha+n}) \\ iQ_{\alpha\beta} &\to -(E_{\alpha,\beta+n} + E_{\beta,\alpha+n}) \\ iK_{\alpha\beta} &\to (E_{\alpha+n,\beta} + E_{\beta+n,\alpha}) \end{aligned} \tag{1.9}$$

where $E_{\mu\nu} \in M_{2n}(\mathbb{R})$ denotes the matrix whose only nonzero entry is one at the intersection of the μ row with the ν column.

An alternative formulation of $bl(n)$ exhibits it as the algebra of skew-adjoint bilinear products of boson operators in n dimensions. Define the boson destruction and creation operators

$$\begin{aligned} a_{j\alpha} &= \frac{1}{\sqrt{2}}(X_{j\alpha} + iP_{j\alpha}) \\ a_{j\alpha}^\dagger &= \frac{1}{\sqrt{2}}(X_{j\alpha} - iP_{j\alpha}) \end{aligned} \tag{1.10}$$

which obey the boson commutation rules

$$[a_{j\alpha}, a_{k\beta}^\dagger] = \delta_{jk}\delta_{\alpha\beta}, \; [a_{j\alpha}, a_{k\beta}] = [a_{j\alpha}^\dagger, a_{k\beta}^\dagger] = 0 \tag{1.11}$$

Then one finds the following correspondence with equation (1.3):

$$iL_{\alpha\beta} = \sum_{j=1}^{N} (a_{j\alpha}^{\dagger} a_{j\beta} - a_{j\beta}^{\dagger} a_{j\alpha})$$

$$iS_{\alpha\beta} = \sum_{j=1}^{N} (a_{j\alpha} a_{j\beta} - a_{j\alpha}^{\dagger} a_{j\beta}^{\dagger})$$

$$iQ_{\alpha\beta} = \frac{i}{2} \sum_{j=1}^{N} (a_{j\alpha}^{\dagger} a_{j\beta} + a_{j\alpha} a_{j\beta}^{\dagger} + a_{j\alpha} a_{j\beta} + a_{j\alpha}^{\dagger} a_{j\beta}^{\dagger}) \tag{1.12}$$

$$iK_{\alpha\beta} = \frac{i}{2} \sum_{j=1}^{N} (a_{j\alpha}^{\dagger} a_{j\beta} + a_{j\alpha} a_{j\beta}^{\dagger} - a_{j\alpha} a_{j\beta} - a_{j\alpha}^{\dagger} a_{j\beta}^{\dagger})$$

For a single particle in one dimension, Itzykson (1967) has given the isomorphism between $sp(1)$ and $bl(1)$ and shown that exp (s) for $s \in bl(1)$ defines a unitary representation of the twofold covering group of $Sp(1)$. Moreover, this representation is the direct sum of two irreducible discrete representations.

Our interest in $sp(n)$ is for its application to N-particle systems (N large) as a collective model for the description of nuclear collective vibrational and rotational states. In one dimension, $sp(1)$ was shown by Goshen and Lipkin (1959) to exhibit vibrational bands for its irreducible discrete series representations. This result was subsequently extended to two-dimensional systems (Goshen and Lipkin, 1968). It is in $sp(3)$ that one expects to have a model capable of explaining both vibrational and rotational bands. Indeed, Biedenharn and Louck (1971) have outlined the possible applicability of $sp(3)$ representations to the problem of extending (dichotomic s-parity) conjugation symmetry (Biedenharn, 1969) to a complete classification scheme for rotational bands of $su(3)$ (Racah, 1964).

The Lie algebra $sp(3)$ is the smallest Lie algebra containing both the subalgebras $su(3)$ and $sl(3)$. Hence, $sp(3)$ gives the minimal algebraic model incorporating both the Elliott $su(3)$ model (Elliott, 1958; Harvey, 1969) and the $sl(3)$ model of Weaver and Biedenharn (1972) for rotational band systems. The algebra $sl(3)$ is the Lie algebra of $SL(3)$, the group of volume-preserving (unit determinant) linear transformations of three-dimensional Euclidean space. The observables spanning $sl(3)$ are the total angular momentum $L_{\alpha\beta}$ and the traceless incompressible stretching (shear) momentum $S_{\alpha\beta}^{(2)}$, cf. equation (1.3),

$$S_{\alpha\beta}^{(2)} = S_{\alpha\beta} - \tfrac{1}{3}\delta_{\alpha\beta}\mathrm{Tr}S \tag{1.13}$$

In Figure 1, we have indicated various subalgebra chains of $sp(3)$. Since $u(3)$ is the symmetry group of the harmonic oscillator, the chain

$$sp(3) > u(3) > su(3) > so(3) \tag{1.14}$$

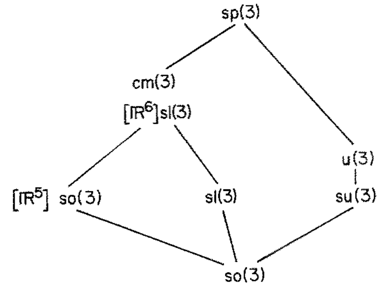

Figure 1. Subalgebra chains of $sp(3)$.

is most appropriate for shell phenomena. On the other hand, both of the chains

$$sp(3) > cm(3) > [\mathbb{R}^5]so(3) > so(3) \tag{1.15}$$

and

$$sp(3) > cm(3) > sl(3) > so(3) \tag{1.16}$$

aim at the description of collective effects. The Lie algebra $cm(3)$ is spanned by the Lie algebra $sl(3)$ and the mass quadrupole moment, $iQ_{\alpha\beta}$, cf. equation (1.3); $cm(3)$ is isomorphic to a semidirect sum $[\mathbb{R}^6]sl(3)$. The $cm(3)$ model is an algebraic formulation of the liquid drop model (Rosensteel and Rowe, 1975b). Moreover, the irreducible representations of $cm(3)$ have been determined (Rosensteel and Rowe, 1976). The semidirect sum $[\mathbb{R}^5]so(3)$ is the Lie algebra spanned by the algebra $so(3)$ together with the traceless mass quadrupole moment,

$$Q^{(2)}_{\alpha\beta} = Q_{\alpha\beta} - \tfrac{1}{3}\delta_{\alpha\beta}\mathrm{Tr}Q \tag{1.17}$$

It has been shown that the irreducible representations of $[\mathbb{R}^5]so(3)$ yield an algebraic model equivalent to the phenomenological rotational model (Ui, 1970; Weaver *et al.*, 1973). Hence, if the nuclear states are chosen symmetry adapted to the chain (1.15), then $E2$ transition rates are predicted in conformity with the rotational model. However, if the chain (1.16) is applied instead, the $sl(3)$ prediction for $E2$ transitions is obtained. Weaver and Biedenharn (1972) have reported that the $sl(3)$ prediction for interband $E2$ transitions is a qualitative improvement on the rotational model.

The plan of this article is to review in Section 2 the discrete series for $Sp(1)$ and its Lie algebra $sp(1)$. In Section 3, the construction for the $Sp(1)$ series is generalized to $Sp(n)$ yielding a discrete series of irreducible unitary representations of $Sp(n)$ with nondegenerate ground states for the harmonic oscillator Hamiltonian. A further generalization of the series of Section 3 is considered in Section 4, which gives representations with degenerate oscillator ground-state eigenspaces irreducible with respect to the action of $U(n)$.

2. *Discrete Series for Sp*(1)

In this section the discrete series for $Sp(1)$ is briefly reviewed. When suitably generalized, the procedure used for $Sp(1)$ yields the construction for the $Sp(n)$ discrete series, cf. Section 3.

A large class of unitary representations of the group $Sp(1) = SL(2, \mathbb{R})$ have been reported and their properties examined by several investigators including Bargmann (1947), Gelfand and Graev (1953), Sally (1967), and Lang (1975). In addition, the skew-adjoint representations of the isomorphic Lie algebras $sp(1) = sl(2, \mathbb{R}) \simeq so(2, 1) \simeq su(1, 1)$ have been given by Bargmann (1947), Barut and Fronsdal (1965), Barut (1965), Barut and Phillips (1968), and Biedenharn et al. (1965). Holman and Biedenharn (1966) have determined the Clebsch-Gordon series for the discrete series. The Lie algebra representations have found many applications in addition to their use for the explanation of vibrational bands in one-dimensional systems (Goshen and Lipkin, 1959). Quesne and Moshinsky (1971) have evaluated the radial integral in the expression for the matrix elements of multipole operators with respect to oscillator states in a one-particle system by the use of $sp(1)$. In this connection, see also Boyer and Wolf (1975). A theory of boson quasispin has been formulated by Ui (1968). Finally, Gambardella (1975) has shown that $su(1,1)$ is the dynamical algebra for that class of many-particle Hamiltonians with a quadratic pair potential plus an arbitrary translation-invariant position-dependent potential homogenous of degree -2. See also Perelomov (1971) and Calogero (1971).

The positive π_w^+ and negative π_w^- discrete series representations for $Sp(1) = SL(2\ \mathbb{R})$ are both indexed by the positive integers $w = 1, 2, 3, \ldots$. The carrier space $\mathscr{H}^2(w)^\pm$ for the positive, or, respectively, negative, discrete series $\pi_w^\pm$ is the space of complex-valued functions f holomorphic, or, respectively, conjugate holomorphic, in the upper half S_1 of the complex plane for which the integral

$$\int_{S_1} \frac{dxdy}{y^2}\, y^w |f(z)|^2 < \infty, \qquad z = x + iy \tag{2.1}$$

is bounded. $\mathscr{H}^2(w)^\pm$ is a Hilbert space with the inner product

$$(f, g) = \int_{S_1} \frac{dxdy}{y^2}\, y^w \overline{f(z)} g(z), \qquad f, g \in \mathscr{H}^2(w)^\pm \tag{2.2}$$

$Sp(1)$ acts on the upper half plane S_1 by

$$z \to M \cdot z \equiv (az + b)(cz + d)^{-1}, \qquad z \in S_1 \tag{2.3}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1)$. The measure on S_1 invariant with respect to this action is

$$d\Omega(z) = dxdy/y^2, \qquad z = x + iy \tag{2.4}$$

Then the positive and negative discrete series are given by the unitary right actions of $Sp(1)$ on $\mathcal{H}^2(w)^\pm$,

$$[\pi_w^+(M)f](z) = (cz + d)^{-w} f(M \cdot z), \qquad f \in \mathcal{H}^2(w)^+ \tag{2.5}$$

$$[\pi_w^-(M)f](z) = (c\bar{z} + d)^{-w} f(M \cdot z), \qquad f \in \mathcal{H}^2(w)^- \tag{2.6}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1)$.

Each unitary right action of $Sp(1)$ defines a skew-adjoint representation of the Lie algebra $sp(1)$ by

$$[\pi_w^\pm(X)f](z) = \frac{d}{d\theta} \{\pi_w^\pm[\exp(-\theta X)]f\}(z)\Big|_{\theta=0}, \qquad f \in \mathcal{H}^2(w)^\pm \tag{2.7}$$

for $X \in sp(1)$. It is most convenient to specify a representation of $sp(1)$ by giving its action on a basis of the complexification of $sp(1)$, viz.,

$$H = \frac{1}{2}(a^\dagger a + aa^\dagger) \to \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$aa \to \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \tag{2.8}$$

and

$$a^\dagger a^\dagger \to \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$$

where

$$H = \tfrac{1}{2} \sum_{j=1}^{N} (P_j^2 + X_j^2)$$

is the harmonic oscillator Hamiltonian, cf. equations (1.3), (1.9), and (1.12). For the positive discrete series, one computes the action to be

$$\pi_w^+(H)f = -i(1 + z^2)\frac{\partial f}{\partial z} - iwzf$$

$$\pi_w^+(a^\dagger a^\dagger)f = i(z - i)^2 \frac{\partial f}{\partial z} + iw(z - i)f \tag{2.9}$$

$$\pi_w^+(aa)f = i(z + i)^2 \frac{\partial f}{\partial z} + iw(z + i)f$$

for $f \in \mathcal{H}^2(w)^+$.

A complete set of eigenstates of the harmonic oscillator Hamiltonian

$$\pi_w^+(H)f_N = Nf_N, \qquad f_N \in \mathscr{H}^2(w)^+ \tag{2.10}$$

is given by

$$f_N(z) = (z - i)^{(N-w)/2}(z + i)^{-(N+w)/2} \tag{2.11}$$

with the spectrum

$$N = w, w + 2, w + 4, \ldots \tag{2.12}$$

due to the condition on the analyticity of f_N in the upper half plane S_1. Then the action of $sp(1)$ on the basis states f_N is given by

$$\begin{aligned} \pi_w^+(a^\dagger a^\dagger)f_N &= -(w + N)f_{N+2} \\ \pi_w^+(aa)f_N &= (w - N)f_{N-2} \end{aligned} \tag{2.13}$$

Since two eigenstates of a self-adjoint operator belonging to different eigenvalues are orthogonal, the vectors f_N and f_M are orthogonal for $N \neq M$. The norm of f_N may be determined by a direct computation using the inner product on S_1, equation (2.2). Alternatively, and more easily, we may normalize the vectors f_N to have a common value, say the norm of the ground state $N_w = \| f_w \|^2$. Thus, we let

$$|N\rangle = C_N f_N \tag{2.14}$$

satisfy

$$\langle N | N \rangle = N_w \tag{2.15}$$

Since

$$\pi_w^+(a^\dagger a^\dagger)|N\rangle = -\frac{C_N}{C_{N+2}}(N + w)|N + 2\rangle \tag{2.16}$$

we have

$$\begin{aligned} \left|\frac{C_N}{C_{N+2}}\right|^2 (N + w)^2 \langle N + 2 | N + 2\rangle &= \langle N | \pi_w^+(aa)\pi_w^+(a^\dagger a^\dagger)|N\rangle \\ &= (N + w)(N - w + 2)\langle N | N\rangle \end{aligned} \tag{2.17}$$

Hence, $\langle N | N\rangle = N_w$ for all $|N\rangle$ if and only if

$$\left|\frac{C_N}{C_{N+2}}\right|^2 = \frac{(N - w + 2)}{(N + w)} \tag{2.18}$$

A solution to this is given by

$$C_N = \sqrt{\frac{(N + w - 2)!!}{(N - w)!!}} \tag{2.19}$$

In summary, the action of the positive discrete series of $sp(1)$ on the orthonormal basis

$$|N\rangle, \qquad N = w, w+2, w+4, \ldots \tag{2.20}$$

is given by

$$\begin{aligned} \pi_w^+(H)|N\rangle &= N|N\rangle \\ \pi_w^+(a^\dagger a^\dagger)|N\rangle &= -\sqrt{N(N+2) - w(w-2)}\,|N+2\rangle \\ \pi_w^+(aa)|N\rangle &= -\sqrt{N(N-2) - w(w-2)}\,|N-2\rangle \end{aligned} \tag{2.21}$$

Required for representations of the group $Sp(1)$, the restriction on w to the positive integers may be lifted for the Lie algebra representations. This may be verified for the action of equation (2.21) by checking directly for all real w that the representation of the real Lie algebra $sp(1)$ is by skew-adjoint operators, which satisfy the relevant commutation relations. The representations of $sp(1)$ with nonintegral w define unitary representations of the universal covering group of $Sp(1)$.

For the negative discrete series π_w^-, the spectrum of the oscillator Hamiltonian is $N = -w - 2r$, $r = 0, 1, 2, \ldots$. The eigenfunction in $\mathscr{H}^2(w)^-$ belonging to the eigenvalue $N = -w - 2r$ is $f_{-N}(z)$. Since the spectrum of the oscillator Hamiltonian for the positive series is bounded from below, it, rather than the negative series, is of most interest for applications.

3. *Discrete Series of* $Sp(1)$: *Nondegenerate Ground States*

The extension to $Sp(n)$ of the discrete series given in Section 2 for $Sp(1)$ requires a generalization of the upper half plane together with an action of $Sp(n)$ to parallel that of equation (2.3). C. L. Siegel (1943) has given the appropriate generalization. Define the Siegel half-plane S_n to be the space of symmetric complex $n \times n$ matrices $z = x + iy$ for which $y = \operatorname{Im} z$ is positive definite. Then $Sp(n)$ acts on S_n by

$$z \to M \cdot z = (az + b)(cz + d)^{-1}, \qquad z \in S_n \tag{3.1}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n)$. Moreover, the measure on S_n invariant with respect to this action is given by

$$d\Omega(z) = \prod_{i \le j} dx_{ij} \wedge \prod_{i \le j} dy_{ij}^{-1} \tag{3.2}$$

where $z = x + iy$ is coordinatized by x_{ij}, $i \le j$, and y_{ij}^{-1}, $i \le j$.

A series of discrete representations $\pi_w^\pm$ of $Sp(n)$ indexed by the positive integers $w = 1, 2, 3, \ldots$ with nondegenerate ground states may now be given. Let the carrier space $\mathscr{H}^2(w)^\pm$ for the representation $\pi_w^\pm$ be the space of complex-valued functions f holomorphic, or, respectively, conjugate holomorphic, in the Siegel half-plane S_n for which the integral

$$\int_{S_n} d\Omega(z)(\det y)^w |f(z)|^2 < \infty, \qquad z = x + iy \tag{3.3}$$

is bounded. $\mathscr{H}^2(w)^\pm$ is a Hilbert space with the inner product

$$(f,g) = \int_{S_n} d\Omega(z)(\det y)^w \overline{f(z)} g(z), \qquad f,g \in \mathscr{H}^2(w)^\pm \tag{3.4}$$

The unitary right action of $Sp(n)$ on $\mathscr{H}^2(w)^\pm$ is given by

$$[\pi_w^+(M)f](z) = \det(cz+d)^{-w} f(M\cdot z), \qquad f \in \mathscr{H}^2(w)^+ \tag{3.5}$$

$$[\pi_w^-(M)f](x) = \det(c\bar{z}+d)^{-w} f(M\cdot z), \qquad f \in \mathscr{H}^2(w)^- \tag{3.6}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n)$. Observe that for $n = 1$ this action is just that given in Section 2, cf. equations (2.1)–(2.6).

A skew-adjoint representation of the Lie algebra $sp(n)$ is defined by

$$[\pi_w^\pm(X)f](z) = \frac{d}{d\theta} \{\pi_w^\pm[\exp(-\theta X)] f\}(z) \Big|_{\theta=0}, \qquad f \in \mathscr{H}^2(w)^\pm \tag{3.7}$$

for $X \in sp(n)$. For the positive series π_w^+ on $\mathscr{H}^2(w)^+$, this representation may be computed on the basis of $sp(n)$ of equation (1.9) to be

$$[\pi_w^+(iL_{\alpha\beta})f](z) = \sum_j \left(z_{\alpha j} \frac{\partial f}{\partial z_{\beta j}} - z_{\beta j} \frac{\partial f}{\partial z_{\alpha j}} \right) + \sum_i \left(z_{i\alpha} \frac{\partial f}{\partial z_{i\beta}} - z_{i\beta} \frac{\partial f}{\partial z_{i\alpha}} \right) \tag{3.8}$$

$$[\pi_w^+(iS_{\alpha\beta})f](z) = -2w\delta_{\alpha\beta} f(z) - \sum_j \left(z_{\alpha j} \frac{\partial f}{\partial z_{\beta j}} + z_{\beta j} \frac{\partial f}{\partial z_{\alpha j}} \right) - \sum_i \left(z_{i\alpha} \frac{\partial f}{\partial z_{i\beta}} + z_{i\beta} \frac{\partial f}{\partial z_{i\alpha}} \right) \tag{3.9}$$

$$[\pi_w^+(iQ_{\alpha\beta})f](z) = \frac{\partial f}{\partial z_{\alpha\beta}} + \frac{\partial f}{\partial z_{\beta\alpha}} \tag{3.10}$$

$$[\pi_w^+(iK_{\alpha\beta})f](z) = w(z_{\alpha\beta} + z_{\beta\alpha}) f(z) + \sum_{ij} (z_{i\alpha} z_{\beta j} + z_{i\beta} z_{\alpha j}) \frac{\partial f}{\partial z_{ij}} \tag{3.11}$$

for $f \in \mathscr{H}^2(w)^+$.

We would like to solve the eigenvalue problem

$$\pi_w^+(iH)f = iNf \tag{3.12}$$

for the harmonic oscillator Hamiltonian,

$$H = \tfrac{1}{2} \sum_\alpha (K_{\alpha\alpha} + Q_{\alpha\alpha}) \tag{3.13}$$

From equations (3.10) and (3.11), this eigenvalue problem is given by

$$w(\mathrm{Tr} z)f(z) + \sum_{ij} (I + z^2)_{ij} \frac{\partial f}{\partial z_{ij}} = iNf \tag{3.14}$$

Every holomorphic solution $f \in \mathcal{H}^2(w)^+$ can be given as a power series in $\zeta = z - iI$ in a neighborhood of $\zeta = 0$,

$$f(\zeta) = C^{(0)} + \sum_{r=1}^{\infty} \sum_{\alpha_1 \leq \beta_1} \cdots \sum_{\alpha_r \leq \beta_r} C^{(r)}_{\alpha_1\beta_1 \cdots \alpha_r\beta_r} \zeta_{\alpha_1\beta_1} \cdots \zeta_{\alpha_r\beta_r} \tag{3.15}$$

with $C^{(r)}_{\alpha_1\beta_1 \cdots \alpha_r\beta_r}$ symmetric under the interchanges $\alpha_i\beta_i \leftrightarrow \alpha_j\beta_j$. Thus $f(\zeta)$ is a solution to equation (3.14) if and only if

$$i(N - nw)C^{(0)} + i \sum_{r=1}^{\infty} (N - nw - 2r) \sum_{\alpha_1 \leq \beta_1} \cdots \sum_{\alpha_r \leq \beta_r} C^{(r)}_{\alpha_1\beta_1 \cdots \alpha_r\beta_r} \zeta_{\alpha_1\beta_1} \cdots \zeta_{\alpha_r\beta_r}$$

$$= w(\mathrm{Tr}\zeta)C^{(0)} + \sum_{r=1}^{\infty} \sum_{\alpha_1 \leq \beta_1} \cdots \sum_{\alpha_r \leq \beta_r} C^{(r)}_{\alpha_1\beta_1 \cdots \alpha_r\beta_r} \left(\sum_{s=1}^{r} \zeta_{\alpha_1\beta_1} \cdots \zeta^2_{\alpha_s\beta_s} \cdots \zeta_{\alpha_r\beta_r} \right.$$

$$\left. + w(\mathrm{Tr}\zeta)\zeta_{\alpha_1\beta_1} \cdots \zeta_{\alpha_r\beta_r} \right) \tag{3.16}$$

Hence, if $N \neq nw, nw + 2, \ldots, nw + 2r$, then $C^{(0)} = C^{(1)} = \cdots = C^{(r)} = 0$. Therefore, the spectrum of $\pi_w^+(H)$ is

$$N = nw + 2r, \qquad r = 0, 1, 2, \ldots$$

An eigenstate belonging to the eigenvalue $N = nw + 2r$ is given by a choice of $C^{(r)}_{\alpha_1\beta_1 \cdots \alpha_r\beta_r}$; the coefficients $C^{(s)}_{\alpha_1\beta_1 \cdots \alpha_s\beta_s}$ for $s > r$ are then defined through equation (3.16). Thus the dimension of the eigenspace belonging to the eigenvalue $N = nw + 2r$ equals the dimension of the space of symmetric r tensors in a space of dimension $m = \frac{1}{2}n(n + 1)$, viz., $\binom{m+r-1}{r}$, see Greub (1967).

In order to explicitly determine the eigenstates, we first simplify the eigenvalue equation (3.12). Let an eigenstate f be written as

$$f(z) = \det(z + iI)^{-w}\Phi(z) \tag{3.17}$$

with $\Phi(z)$ analytic in S_n. Then f is an eigenstate belonging to the eigenvalue $N = nw + 2r$ if and only if Φ satisfies

$$\sum_{ij} (I + z^2)_{ij} \frac{\partial \Phi}{\partial z_{ij}} = 2ir\Phi(z) \tag{3.18}$$

since for any invertible matrix g_{ij}

$$\frac{\partial}{\partial g_{ij}} (\det g) = (\det g)(g^{-1})_{ij} \tag{3.19}$$

The solutions to the eigenvalue problem for Φ belonging to the eigenvalue $r = 1$ are

$$\Phi_{\mu\nu}(z) = [(z - iI)(z + iI)^{-1}]_{\mu\nu} \tag{3.20}$$

as can be shown by expanding $\Phi_{\mu\nu}$ in a power series in $\zeta = z - iI$ about $\zeta = 0$. Since the operator $\Sigma_{ij}(I + z^2)_{ij}\partial/\partial z_{ij}$ is a derivation, eigenfunctions Φ belonging to integral r eigenvalues are given by the product functions,

$$\Phi_{\mu_1\nu_1}(z)\Phi_{\mu_2\nu_2}(z)\cdots\Phi_{\mu_r\nu_r}(z) \tag{3.21}$$

Hence, a complete set of eigenstates for the harmonic oscillator $\pi_w^+(H)$ belonging to the eigenvalue $N = nw + 2r$ is given by

$$f_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_r\nu_r}(z) = \det(z + iI)^{-w}\Phi_{\mu_1\nu_1}(z)\Phi_{\mu_2\nu_2}(z)\cdots\Phi_{\mu_r\nu_r}(z) \tag{3.22}$$

since the number of independent solutions $f_{\mu_1\nu_1\cdots\mu_r\nu_r}$ equals the dimension of the space of symmetric r tensors in a space of dimension $m = \frac{1}{2}n(n + 1)$.

The action of the Lie algebra $sp(n)$ on the basis states $f_{\mu_1\nu_1\cdots\mu_r\nu_r}$ may be computed to be

$$\begin{aligned}
\pi_w^+(a_\alpha^\dagger a_\beta^\dagger)f_{\mu_1\nu_1\cdots\mu_r\nu_r} &= -2wf_{\mu_1\nu_1\cdots\mu_r\nu_r\alpha\beta} \\
&\quad - \sum_{s=1}^{r}(f_{\mu_1\nu_1\cdots\mu_s\alpha\cdots\mu_r\nu_r\beta\nu_s} + f_{\mu_1\nu_1\cdots\mu_s\beta\cdots\mu_r\nu_r\alpha\nu_s}) \\
\pi_w^+(a_\alpha a_\beta)f_{\mu_1\nu_1\cdots\mu_r\nu_r} &= -\sum_{s=1}^{r}(\delta_{\mu_s\alpha}\delta_{\beta\nu_s} + \delta_{\mu_s\beta}\delta_{\alpha\nu_s})f_{\mu_1\nu_1\cdots\widehat{\mu_s\nu_s}\cdots\mu_r\nu_r}
\end{aligned} \tag{3.23}$$

where

$$f_{\mu_1\nu_1\cdots\widehat{\mu_s\nu_s}\cdots\mu_r\nu_r} \equiv f_{\mu_1\nu_1\cdots\mu_{s-1}\nu_{s-1}\mu_{s+1}\nu_{s+1}\cdots\mu_r\nu_r}$$

$$\begin{aligned}
\pi_w^+(a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger)f_{\mu_1\nu_1\cdots\mu_r\nu_r} &= 2w\delta_{\alpha\beta}f_{\mu_1\nu_1\cdots\mu_r\nu_r} \\
&\quad + 2\sum_{s=1}^{r}(\delta_{\beta\nu_s}f_{\mu_1\nu_1\cdots\mu_s\alpha\cdots\mu_r\nu_r} + \delta_{\beta\mu_s}f_{\mu_1\nu_1\cdots\alpha\nu_s\cdots\mu_r\nu_r})
\end{aligned}$$

The unitary group $U(n)$ is imbedded in $Sp(n)$ through the homomorphism

$$\chi = U + iV \in U(n) \to \begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in Sp(n) \tag{3.24}$$

Then the representation of $U(n)$ is given by

$$\begin{aligned}
\pi_w^+(\chi)f_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_r\nu_r} &= (\det\chi)^w \sum_{\mu_1'\nu_1'}\cdots\sum_{\mu_r'\nu_r'} f_{\mu_1'\nu_1'\cdots\mu_r'\nu_r'}\chi_{\mu_1'\mu_1}\chi_{\nu_1'\nu_1}\cdots \\
&\quad \times \chi_{\mu_r'\mu_r}\chi_{\nu_r'\nu_r}
\end{aligned} \tag{3.25}$$

for $\chi \in U(n)$.

In particular, the action of $SU(n)$ on the first excited level $N = nw + 2$ is

$$\pi_w^+(\chi)f_{\mu_1\nu_1} = \sum_{\mu_1'\nu_1'} f_{\mu_1'\nu_1'}\chi_{\mu_1'\mu_1}\chi_{\nu_1'\nu_1} \tag{3.26}$$

which is equivalent to the irreducible representation $(2, 0, \ldots, 0)$ of $SU(n)$ (Hamermesh, 1962). Hence, the action of $SU(n)$ on the rth level $N = nw + 2r$ is equivalent to the (reducible) symmetric tensor product representation of r copies of the $(2, 0, \ldots, 0)$ representation of $SU(n)$.

In Figure 2a, the first few oscillator levels for the π_w^+ representation of $Sp(3)$ are indicated together with the irreducible representations (λ, μ) of $SU(3)$ spanning each degenerate level. In addition, the angular momenta L of each level are given. This angular momentum spectrum is similar to that of the collective (liquid-drop) vibrational model of Bohr (1952). In the Bohr model, the equally spaced levels are given by the symmetric tensor product representation of $SO(3)$ built upon the $L = 2$ representation; the angular momenta of the first few levels are given in Figure 2b, see Hecht (1964). The

(λ, μ)	N	L
(6, 0) (2, 2) (0, 0)	$3w + 6$	$0^3, 2^3, 3, 4^2, 6$
(4, 0) (0, 2)	$3w + 4$	$0^2, 2^2, 4$
(2, 0)	$3w + 2$	0, 2
(0, 0)	$3w$	0

(a)

N	L
6	0, 2, 3, 4, 6
4	0, 2, 4
2	2
0	0

(b)

Figure 2. (a) In the $Sp(3)$ model, the first four oscillator levels are indicated. The $SU(3)$ irreps (λ, μ) and angular momenta L spanning each level are given. (b) In the Bohr model, the angular momenta L of the oscillator levels are given.

evident difference between the Bohr model and the $Sp(3)$ model for collective vibrations lies in the additional monopole $L = 0$ excitation of the first excited level in the $Sp(3)$ model.

In order to determine the inner product in $\mathscr{H}^2(w)^+$, it is most practical to use a method similar to that employed in Section 2 for the $Sp(1)$ representations rather than to attempt a direct computation from equation (3.4). For $Sp(3)$ one first chooses an orthogonal basis $|N(\lambda\mu)KLM\rangle$ symmetry-adapted to the subgroup chain

$$Sp(3) > U(3) > SU(3) > SO(3) > SO(2) \tag{3.27}$$

This basis may be computed using the fractional parentage and Clebsch-Gordon coefficients for $SU(3)$ (Hecht, 1965). Moreover, the norm of the states $|N(\lambda\mu)KLM\rangle$ carrying a given $(\lambda\mu)$ representation of $SU(3)$ may be fixed to a common value, say the norm of the highest weight state denoted $\Phi[N(\lambda\mu)]$. Finally, the norms of the highest weight states are made equal to the norm of the ground state by the following inductive procedure: If the norms of the vectors in the $N = nw + 2r$ level equal that of the ground state, then the norm of $\Phi[N + 2(\lambda\mu)]$ is given by

$$\begin{aligned}&(\phi[N + 2(\lambda\mu)], \pi_w^+(s^\dagger)\pi_w^+(s)\phi[N + 2(\lambda\mu)])\\&= (\pi_w^+(s)\phi[N + 2(\lambda\mu)], \pi_w^+(s)\phi[N + 2(\lambda\mu)])\end{aligned} \tag{3.28}$$

for any $s \in bl(3)$, quanta-annihilating say $s = \Sigma_\alpha a_\alpha a_\alpha$, for which the right-hand side of equation (3.28) is nonzero.

We close this section with the remark that the restriction to integral values for w, while necessary for group representations of $Sp(n)$, is not required for the Lie algebra representations of $sp(n)$, equation (3.23).

4. *Discrete Series of Sp(n): The General Case*

The discrete series of irreducible unitary representations of $Sp(n)$ with non-degenerate ground states given in Section 3 possesses a further generalization due to R. Godement (1958). Let ρ be an irreducible tensor representation of $GL_+(n)$ carried by a finite-dimensional vector space F, where $GL_+(n)$ denotes the group of $n \times n$ real matrices with positive determinant (Weyl, 1946). Then Godement has given a positive and negative discrete representation π_ρ^+ of $Sp(n)$ for each such ρ that is unitary and irreducible.

The carrier space $\mathscr{H}^2(\rho)^\pm$ for the representation $\pi_\rho^\pm$ is the space of functions f taking values in F holomorphic, or, respectively, conjugate holomorphic, in the Siegel half-plane S_n for which the integral

$$\int_{S_n} d\Omega(z) \| \rho(y^{1/2}) f(z) \|_F^2 < \infty, \qquad z = x + iy \tag{4.1}$$

is bounded. $\mathscr{H}^2(\rho)^\pm$ is a Hilbert space with the inner product

$$(f, g) = \int_{S_n} d\Omega(z) (\rho(y^{1/2}) f(z), \rho(y^{1/2}) g(z))_F, \qquad f, g \in \mathscr{H}^2(\rho)^\pm \tag{4.2}$$

The irreducible unitary right action of $Sp(n)$ on $\mathscr{H}^2(\rho)^\pm$ is given by

$$[\pi_\rho^+(M) f](z) = \rho(cz + d)^{-1} f(M \cdot z), \qquad f \in \mathscr{H}^2(\rho)^+ \tag{4.3}$$

$$[\pi_\rho^-(M) f](z) = \rho(c\bar{z} + d)^{-1} f(M \cdot z), \qquad f \in \mathscr{H}^2(\rho)^- \tag{4.4}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n)$. Observe that the representations of Section 3 are given by the one-dimensional representations of $GL_+(n)$, $\rho(g) = (\det g)^w$.

For the skew-adjoint representation of the Lie algebra $sp(n)$, also denoted by $\pi_\rho^\pm$, we wish to solve the eigenvalue problem for the harmonic oscillator Hamiltonian H.

A rank l tensor representation $\rho^{[\lambda]}$ of $GL_+(n)$ is determined by an n-tuple of positive integers $[\lambda] = [\lambda_1, \lambda_2, \ldots \lambda_n]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$ such that

$$\sum_{i=1}^{n} \lambda_i = l$$

The representation $\rho^{[\lambda]}$ is irreducible and carried by a finite-dimensional vector subspace, denoted $F^{[\lambda]}$, of the tensor product space $\overset{l}{\otimes}\, \mathbb{C}^n \equiv \mathbb{C}^n \otimes \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$ (l copies). In fact, $F^{[\lambda]}$ is a subspace of $\overset{l}{\otimes}\, \mathbb{C}^n$ irreducible with respect to the action of the permutation group on l symbols $\mathscr{S}_l$ given by

$$e_1 \otimes e_2 \otimes \cdots \otimes e_l \xrightarrow{p} e_{p(1)} \otimes e_{p(2)} \otimes \cdots \otimes e_{p(l)} \tag{4.5}$$

for $p \in \mathscr{S}_l$ and

$$\overset{l}{\otimes}\, \mathbb{C}^n \simeq \bigoplus_{[\lambda]} F^{[\lambda]} \tag{4.6}$$

The irreducible representation $\rho^{[\lambda]}$ is the restriction to $F^{[\lambda]}$ of the rank l tensor representation ρ^l of $GL_+(n)$ on $\overset{l}{\otimes}\mathbb{C}^n$ given by

$$\rho^l(g)(e_{\sigma_1} \otimes e_{\sigma_2} \otimes \cdots \otimes e_{\sigma_l}) = \sum_{\tau_1\tau_2 \cdots \tau_l} (e_{\tau_1} \otimes e_{\tau_2} \otimes \cdots \otimes e_{\tau_l}) g_{\tau_1\sigma_1} g_{\tau_2\sigma_2} \cdots g_{\tau_l\sigma_l} \tag{4.7}$$

for $g \in GL_+(n)$. Hence, it is evident that $\mathscr{H}^2(\rho^l)^\pm \simeq \bigoplus_{[\lambda]} \mathscr{H}^2(\rho^{[\lambda]})^\pm$ and $\pi^\pm_{\rho^l} \simeq \bigoplus_{[\lambda]} \pi^\pm_{\rho^{[\lambda]}}$. We shall determine the spectrum and eigenstates for $\pi^\pm_{\rho^l}(H)$ in $\mathscr{H}^2(\rho^l)^\pm$; the solutions to the eigenvalue problem for $\pi^\pm_{\rho^{[\lambda]}}$ are then given by the restriction to the irreducible subspace $\mathscr{H}^2(\rho^{[\lambda]})^\pm$.

If $\pi^+_{\rho^l}(H)f = Nf$ and $f \in \mathscr{H}^2(\rho^l)^+$ is given by

$$f(z) = \sum_{\sigma_1\sigma_2 \cdots \sigma_l} f_{\sigma_1\sigma_2 \cdots \sigma_l}(z) e_{\sigma_1} \otimes e_{\sigma_2} \otimes \cdots \otimes e_{\sigma_l} \tag{4.8}$$

then

$$\sum_{k=1}^{l} \left[\sum_{\tau_k} z_{\sigma_k\tau_k} f_{\sigma_1\sigma_2 \cdots \tau_k \cdots \sigma_l}(z) \right] + \sum_{ij} (I + z^2)_{ij} \frac{\partial f_{\sigma_1\sigma_2 \cdots \sigma_l}}{\partial z_{ij}} = iNf_{\sigma_1\sigma_2 \cdots \sigma_l}(z) \tag{4.9}$$

If $f_{\sigma_1\sigma_2 \cdots \sigma_l}(z)$ is expanded in a power series in $\zeta = z - iI$, an equation for the coefficients of the expansion is obtained as in equation (3.16) of Section 3. As a result it is deduced that the spectrum of $\pi^+_{\rho^l}(H)$ is

$$N = l + 2r, \qquad r = 0, 1, 2, \ldots \tag{4.10}$$

Moreover, the dimension of the eigenspace in $\mathscr{H}^2(\rho^l)^+$ belonging to the eigenvalue $N = l + 2r$ equals $n^l\binom{m+r-1}{r}$.

For each l-tuple of integers $(\tau_1\tau_2 \cdots \tau_l)$, $1 \le \tau_k \le n$, consider the vector in $\mathscr{H}^2(\rho^l)^+$,

$$f^{(\tau_1\tau_2 \cdots \tau_l)} = \sum_{\sigma_1\sigma_2 \cdots \sigma_l} \otimes e_{\sigma_1} \otimes e_{\sigma_2} \cdots \otimes e_{\sigma_l} (z + iI)^{-1}_{\sigma_1\tau_1} (z + iI)^{-1}_{\sigma_2\tau_2} \cdots \times (z + iI)^{-1}_{\sigma_l\tau_l} \Phi(z) \tag{4.11}$$

where $\Phi(z)$ is complex-valued and analytic in S_n. Then $f^{(\tau_1\tau_2 \cdots \tau_l)}$ is an eigenvector of $\pi^+_{\rho^l}(H)$ belonging to the eigenvalue $N = l + 2r$ if and only if Φ satisfies

$$\sum_{ij} (I + z^2)_{ij} \frac{\partial \Phi}{\partial z_{ij}} = 2ir\Phi(z) \tag{4.12}$$

But the solutions $\Phi(z)$ to this equation were given in Section 3 in equations (3.20) and (3.21). Hence a complete set of eigenstates of $\pi_\rho^+ \iota(H)$ in $\mathscr{H}^2(\rho^l)^+$ is given by

$$f^{(\tau_1\tau_2\cdots\tau_l)}_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_r\nu_r}(z) = \sum_{\sigma_1\sigma_2\cdots\sigma_l} e_{\sigma_1}\otimes e_{\sigma_2}\otimes\cdots\otimes e_{\sigma_l}(z+iI)^{-1}_{\sigma_1\tau_1}(z+iI)^{-1}_{\sigma_2\tau_2}\cdots (z+iI)^{-1}_{\sigma_l\tau_l}\cdots\Phi_{\mu_1\nu_1}(z)\Phi_{\mu_2\nu_2}(z)\cdots\Phi_{\mu_r\nu_r}(z) \qquad (4.13)$$

The action of the subgroup $U(n)$ is given by

$$\pi_\rho^+\iota(\chi)f^{(\tau_1\cdots\tau_l)}_{\mu_1\nu_1\cdots\mu_r\nu_r} = \sum_{\tau_1'\cdots\tau_l'}\sum_{\mu_1'\nu_1'}\cdots\sum_{\mu_r'\nu_r'} f^{(\tau_1'\cdots\tau_l')}_{\mu_1'\nu_1'\cdots\mu_r'\nu_r'} \times \chi_{\mu_1'\mu_1}\chi_{\nu_1'\nu_1}\cdots\chi_{\mu_r'\mu_r}\chi_{\nu_r'\nu_r}\cdots\chi_{\tau_1'\tau_1}\cdots\chi_{\tau_l'\tau_l} \qquad (4.14)$$

for $\chi\in U(n)$. Restricted to the irreducible subspace $\mathscr{H}^2(\rho^{[\lambda]})^+$, it is clear that the ground state transforms according to the $[\lambda]$ irreducible representation of $U(n)$. In particular, the representation π_w^+ of Section 3 is the restriction to the subspace defined by $[w, w, \ldots., w]$.

We have thus found that for every irreducible representation $[\lambda]$ of $U(n)$ there is a discrete irreducible unitary representation of $Sp(n)$ for which the oscillator Hamiltonian is bounded from below with its ground-state eigenspace transforming irreducibly under $U(n)$ according to $[\lambda]$.

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